Generalization of Meromorphic Function

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Abstract: A complex valued function is different from a real valued function in certain properties and the present discussion is regarding the factorization of a complex function completely which is not possible for the real functions. The factorization of the rational function $f(z)=\frac{p(z)}{q(z)}$ having the representation of the denominator $q(z)=d\left(z-b_{q_1}\right)^{n_1}\left(z-b_{q_2}\right)^{n_2}\ldots\left(z-b_{q_s}\right)^{n_s}$ with d to be a constant, the points $b_{q_j}, 1\le j\le s$ are the singularities which are proved to be isolated by showing that each of these singularities have empty neighbourhoods. This enables us, there are no patches of non existence of complex function in the complex plane except at isolated points. This leads to the strong result that every complex valued function is either an entire function or a meromorphic function.

1. ***Introduction***: If $f(x)$ is a real valued function such that $f(a)=0$, then a is the zero of f and $x-a$ is a factor of f with multiplicity respectively [1]. But every real valued function is not necessarily to have a factor. It is observed that a real valued function $f(x)=\frac{1}{\sqrt{1-x^2}}$ is not defined on the entire real line except in (-1, 1). That means, this function has singularity at every point of $\mathbb{R}-(-1,1)$. In particular, every real number in the neighbourhood of 2 is a singularity of f. This shows that, 2 is not an isolated singularity of $f(x)=\frac{1}{\sqrt{1-x^2}}$. Similarly, every real number in $\mathbb{R}-(-1,1)$ is not an isolated singularity. On the other hand, $f(z)=\frac{1}{\sqrt{1-z^2}}$, $z\in\mathbb{C}$(complex plane) is defined at every complex number except $z=-1$ and $z=1$. With a careful observation, there is no other complex number in the neighbourhood of $z=-1$ is a singularity of $f(z)=\frac{1}{\sqrt{1-z^2}}$. Similarly, there is no other complex number in the neighbourhood of $z=1$ which will be a singularity of this function. Therefore, $z=-1$ and $z=1$ are the isolated singularities of $f(z)=\frac{1}{\sqrt{1-z^2}}$. Every complex valued function has a polynomial representation even though it is not analytic at certain points. This polynomial representation can have negative exponents of z which shows that the given complex valued function has singularities where the function is not defined and so, not analytic. On the other hand, the complex function analytic in the domains not having singularities will have polynomial representation only with non negative powers.
If a complex function $f(z)$ takes the rational form, then the zeroes of the denominator are the singularities of f. A zero of multiplicity 1 is a simple zero.

Definition 1.1: a complex valued function $f(z)$ is continuous at a point $z=a$ if to each $\varepsilon>0$, there corresponds a $\delta>0$ such that $|z-a|<\delta$ implies $|f(z)-f(a)|<\varepsilon$

Definition 1.2: a complex valued function $f(z)$ is differentiable at a point $z=a$, if $\lim_{z\to a}\frac{f(z)-f(a)}{z-a}$ exists and it is denoted by $f'(a)$.

Definition 1.3: a continuously differentiable function is said to be an analytic function. That is, if $\lim_{z_1\to z_2}\frac{f(z_1)-f(z_2)}{z_1-z_2}$ exists at every pair of $z_1,z_2\in B(a,R)$ with $|z_1-z_2|<\delta$

Definition 1.4: if a complex valued function $f(z)$ is analytic in the entire complex plane $\mathbb{C}$, then it is called an entire function.[2]&[4].

Definition 1.5: If a complex valued function *f*(*z*) is an analytic function in the entire complex plane ℂ except at its poles is a meromorphic function.[3]&[5]

Chapter 2:

Fundamental theorem of Algebra 2.1: either a complex valued function is a constant or it can be written as the product of linear factors and a constant.

That is, $f(z) = k(z-a_1)^{m_1}(z-a_2)^{m_2} \ldots$ where $a_i{}'$ s are the zeroes of *f* and each zero is repeated for m_i times. The zeroes may be either finite or infinite in number while *k* is a constant. [7]&[8].

Theorem 2.2: The singularities of a complex valued function are isolated singularities.

Proof: without loss of generality, suppose *G* = *B* (*a*, *R*) is an open set in which the complex valued function *f* (*z*) has the radius of convergence *R*.

The complex valued function *f*(*z*) has singularities if and only if it is a rational function of the form $\frac{p(z)}{q(z)}$ where both *p*(*z*) and *q*(*z*) have the polynomial representations and in particular, the polynomial of *q*(*z*) will be centred at the singularities of *f*(*z*).

By the fundamental theorem of algebra, either *q*(*z*) is a constant or it can be written as the product of linear factors with the respective multiplicities.

That is, $p(z) = c\left(z-a_{p_1}\right)^{m_1}\left(z-a_{p_2}\right)^{m_2} \ldots \left(z-a_{p_k}\right)^{m_k}$ and

$q(z) = d\left(z-b_{q_1}\right)^{n_1}\left(z-b_{q_2}\right)^{n_2} \ldots \left(z-b_{q_s}\right)^{n_s}$ with *c* and *d* constants, $a_{p_i,} 1 \le i \le k, b_{q_j}, 1 \le j \le s$ are the zeroes of *p*(*z*) and *q*(*z*) respectively with the respective multiplicities $m_i, 1 \le i \le k$ and $n_j, 1 \le j \le s$.

But, $b_{q_j}, 1 \le j \le s$ are the singularities of *f*(*z*).

We now try to prove that b_{q_j} is an isolated singularity of *f* in *G* for each $1 \le j \le s$

For, assume that Minimum of $\left|b_{q_r} - b_{q_t}\right| = \delta, 1 \le r, t \le s$ (2.3)

Recollect a theorem that states 'if *q* is an analytic function in *G* = *B* (b_{q_r}, *R*) with *q* (b_{q_r}) = α and *q* (*z*) – α has the zero b_{q_r} of multiplicity *m*, then *q* (*z*) – β has *m* simple zeroes of the form b_{q_t} such that $|\alpha - \beta| < \varepsilon$ where δ and ε are dependent.

In view of this theorem, b_{q_r} is a zero of *q* (*z*) – α having multiplicity *m* while b_{q_t} is a simple zero of *q* (*z*) – β which establishes that $b_{q_r} \ne b_{q_t}$ whenever $r \ne t$.

Using (2.3), it follows $b_{q_r} \ne b_{q_t}$ for all $1 \le r \ne t \le s$.

Or, any two singularities of *f* are at a minimum distance of $\delta > 0$

In other words, any two singularities are isolated singularities for *f*(*z*).

Consequence 2.4: every complex valued function is either an entire function or a meromorphic function.

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